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On a Difference Equation for Generalizations of Charlier Polynomials

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In this paper we obtain a set of polynomials which are orthogonal with respect to the classical discrete weight function of the Charlier polynomials at which an extra point mass at $x=0$ is added. We construct a difference operator of infinite order for which these new discrete orthogonal polynomials are eigenfunctions.

1. INTRODUCTION

In [6] J. Koekoek and R. Koekoek found a differential equation of the form

$$N \sum_{i=0}^{\infty} a_i(x)\, y^{(i)}(x) + xy''(x) + (\alpha+1-x)\, y'(x) + ny(x) = 0$$

for the polynomials $\{L_n^{\alpha,N}(x)\}_{n=0}^{\infty}$, which are orthogonal on the interval $[0, \infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x} + N\delta(x), \qquad \alpha > -1, N \geqslant 0.$$

The coefficients $\{\alpha_i(x)\}_{i=1}^{\infty}$ are independent of the degree n and $a_0(x)$ is independent of x. When $N>0$ this differential equation is of infinite order in general and for nonnegative integer values of the parameter α the order reduces to $2\alpha+4$.

In [7] R. Koekoek also found a similar differential equation for the

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symmetric generalized ultraspherical polynomials $\{P_n^{\alpha,\alpha,M,M}(x)\}_{n=0}^{\infty}$, which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\{\Gamma(\alpha+1)\}^2}(1-x^2)^\alpha + M[\delta(x+1)+\delta(x-1)], \qquad \alpha > -1, M \geqslant 0.$$

For more details concerning these generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ the reader is referred to [8].

In [4] R. A. Askey posed the problem of finding difference equations of a similar form for generalizations of discrete orthogonal polynomials which are orthogonal with respect to the classical weight function together with an extra point mass at the point $x=0$.

In this paper we solve this problem for generalizations of the classical Charlier polynomials.

In fact, we look for difference equations of the form

$$N\sum_{i=0}^{\infty} A_i(x)\,\Delta^i y(x) + x\Delta\nabla y(x) + (a-x)\,\Delta y(x) + ny(x) = 0 \qquad (1)$$

satisfied by the polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to the inner product

$$\langle f, g\rangle = \sum_{x=0}^{\infty} \frac{e^{-a}a^x}{x!} f(x)\,g(x) + Nf(0)\,g(0), \qquad a>0, N\geqslant 0.$$

In this paper we give a constructive method for obtaining the coefficients $\{A_i(x)\}_{i=0}^{\infty}$ in the difference equation (1) and we show that if $N>0$ the order of this difference equation turns out to be infinite for all values of the parameter $a>0$.

In [3] the similar problem for generalizations of the Meixner polynomials is treated.

In [1] H. Bavinck introduced Sobolev-type generalizations of the Charlier polynomials and in [2] it is shown that these are eigenfunctions of a difference operator of infinite order as well.

2. Definitions and Notations

We will use the following definition of the classical Charlier polynomials:

$$C_n^{(a)}(x) := \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!} = \frac{(-a)^n}{n!}\,{}_2F_0\left(\begin{matrix}-n, -x\\ -\end{matrix}\,\middle|\, -\frac{1}{a}\right), \quad n=0, 1, 2, \ldots. \qquad (2)$$

This definition is slightly different from the one given in [5] but this one turns out to be very convenient in this work. For further details concerning the classical Charlier polynomials, the reader is referred to [5] anyway.

We remark that we have

$$C_n^{(a)}(x) = L_n^{(x-n)}(a), \qquad n = 0, 1, 2, \ldots, \tag{3}$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomial defined by

$$L_n^{(\alpha)}(x) := \frac{1}{n!} \sum_{k=0}^{n} (-n)_k \, (\alpha + k + 1)_{n-k} \frac{x^k}{k!}, \; n = 0, 1, 2, \ldots.$$

Further, we define the difference operators

$$\Delta f(x) := f(x+1) - f(x) \tag{4}$$

and

$$\nabla f(x) := f(x) - f(x-1). \tag{5}$$

Then we find

$$\Delta C_n^{(a)}(x) = C_{n-1}^{(a)}(x), \qquad n = 1, 2, 3, \ldots. \tag{6}$$

We also have

$$C_n^{(a)}(0) = \frac{(-a)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

and

$$C_n^{(a)}(-1) = (-1)^n e_n^a, \quad n = 0, 1, 2, \ldots,$$

where

$$e_n^a := \sum_{k=0}^{n} \frac{a^k}{k!}.$$

The classical Charlier polynomials are discrete orthogonal polynomials which satisfy the orthogonality relation given by

$$\sum_{x=0}^{\infty} \frac{e^{-a} a^x}{x!} C_m^{(a)}(x) \, C_n^{(a)}(x) = \frac{a^n}{n!} \delta_{mn}, \qquad a > 0, m, n = 0, 1, 2, \ldots,$$

where δ_{mn} denotes the Kronecker delta.

They satisfy a second order difference equation which can be written in the form

$$x\Delta\,\nabla y(x)+(a-x)\,\Delta y(x)+ny(x)=0, \qquad y(x)=C_n^{(a)}(x).$$

By using the definition of the difference operators (4) and (5) we may rewrite this as

$$ay(x+1)+(n-a-x)\,y(x)+xy(x-1)=0, \qquad y(x)=C_n^{(a)}(x). \tag{7}$$

From the generating function

$$e^{-at}(1+t)^x=\sum_{n=0}^{\infty} C_n^{(a)}(x)\,t^n$$

we easily obtain

$$\sum_{m=0}^{\infty} C_m^{(a)}(x)\,t^m \sum_{j=0}^{\infty} C_j^{(-a)}(-x)\,t^j=e^{-at}(1+t)^x\,e^{at}(1+t)^{-x}=1.$$

Hence

$$\sum_{m=0}^{k} C_m^{(a)}(x)\,C_{k-m}^{(-a)}(-x)=\begin{cases}1, & k=0,\\ 0, & k=1,2,3,\dots.\end{cases}$$

This can also be written as

$$\sum_{k=j}^{i} C_{i-k}^{(a)}(x)\,C_{k-j}^{(-a)}(-x)=\delta_{ij}, \qquad j\leqslant i,\ i,j=0,1,2,\dots. \tag{8}$$

Formula (8) plays an important role in Sections 4 and 5 of this paper.

We will use another elegant formula which can be obtained from the generating function. We have for arbitrary real p

$$\sum_{n=0}^{\infty} C_n^{(a)}(x+p)\,t^n=e^{-at}(1+t)^{x+p}=(1+t)^p\sum_{m=0}^{\infty} C_m^{(a)}(x)\,t^m.$$

Hence

$$C_n^{(a)}(x+p)=\sum_{k=0}^{n}\binom{p}{k}C_{n-k}^{(a)}(x), \qquad n=0,1,2,\dots.$$

The special case $p=-1$ reads

$$C_n^{(a)}(x-1)=\sum_{k=0}^{n}(-1)^k\,C_{n-k}^{(a)}(x), \qquad n=0,1,2,\dots. \tag{9}$$

The special case $p=n$ can be written as

$$C_n^{(a)}(x+n)=\sum_{k=0}^{n}\binom{n}{k}C_k^{(a)}(x), \qquad n=0,1,2,.... \tag{10}$$

3. Generalizations of the Charlier Polynomials

Let P denote the space of all real polynomials with real coefficients. In this section we will determine a set of polynomials which are orthogonal with respect to the inner product

$$\langle f,g\rangle=\sum_{x=0}^{\infty}\frac{e^{-a}a^x}{x!}f(x)\,g(x)+Nf(0)\,g(0), \qquad a>0, N>0, \text{ and } f,g\in P. \tag{11}$$

If we denote this set of polynomials by $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$, where degree $[C_n^{a,N}(x)]=n$, then we will show that coefficients A_n and B_n can be determined in such a way that these polynomials can be written in the form

$$C_n^{a,N}(x)=A_nC_n^{(a)}(x)+B_nC_n^{(a)}(x-1).$$

Suppose that $n\geqslant 2$ and

$$p(x)=xq(x) \qquad \text{with} \quad \text{degree}[q(x)]\leqslant n-2.$$

Then we easily obtain by using the orthogonality property of the classical Charlier polynomials

$$\begin{aligned}\langle p(x),C_n^{a,N}(x)\rangle&=B_n\sum_{x=0}^{\infty}\frac{e^{-a}a^x}{x!}xq(x)\,C_n^{(a)}(x-1)\\&=aB_n\sum_{x=0}^{\infty}\frac{e^{-a}a^x}{x!}q(x+1)\,C_n^{(a)}(x)=0.\end{aligned}$$

Hence, A_n and B_n must satify for $n\geqslant 1$

$$0=\langle 1,C_n^{a,N}(x)\rangle=B_n\sum_{x=0}^{\infty}\frac{e^{-a}a^x}{x!}C_n^{(a)}(x-1)+NA_nC_n^{(a)}(0)+NB_nC_n^{(a)}(-1).$$

Now we use (9) and the orthogonality property of the classical Charlier polynomials to obtain

$$\sum_{x=0}^{\infty}\frac{e^{-a}a^x}{x!}C_n^{(a)}(x-1)=(-1)^n.$$

Hence,

$$NA_n C_n^{(a)}(0) + [(-1)^n + NC_n^{(a)}(-1)]\, B_n = 0, \qquad n = 1, 2, 3, \ldots.$$

So we may choose

$$A_n = 1 + N(-1)^n\, C_n^{(a)}(-1)$$

and

$$B_n = N(-1)^{n-1}\, C_n^{(a)}(0), \qquad n = 0, 1, 2, \ldots,$$

which leads to the following proposition.

PROPOSITION. *The generalized Charlier polynomials* $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ *which are orthogonal with respect to the inner product* (11) *can be defined by*

$$\begin{aligned} C_n^{a,N}(x) &= [1 + N(-1)^n\, C_n^{(a)}(-1)]\, C_n^{(a)}(x) \\ &\quad - N(-1)^n\, C_n^{(a)}(0)\, C_n^{(a)}(x-1), \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{12}$$

Note that we have chosen $C_0^{a,N}(x) = C_0^{(a)}(x) = 1$. Further we remark that we can write

$$\begin{aligned} C_n^{a,N}(x) &= [1 + N(-1)^{n-1}\, C_{n-1}^{(a)}(-1)]\, C_n^{(a)}(x) \\ &\quad + N(-1)^n\, C_n^{(a)}(0)\, \Delta C_n^{(a)}(x-1), \end{aligned}$$

for $n \geqslant 1$ since we have

$$C_n^{(a)}(0) - C_n^{(a)}(-1) = C_{n-1}^{(a)}(-1), \qquad n = 1, 2, 3, \ldots.$$

For convenience we define $C_{-1}^{(a)}(x) \equiv 0$ in the sequel. Now the latter definition holds for all $n \in \{0, 1, 2, \ldots\}$.

4. The Difference Equation

We try to find a difference equation of the form (1) for the polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ found in the preceding section and given by (12), where the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are arbitrary functions of x independent of the degree n. Since we want the polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ to be eigenfunctions of a difference operator we assume that $A_0(x)$ does not depend on x.

So we set

$$\begin{aligned} y(x) = C_n^{a,N}(x) &= [1 + N(-1)^n\, C_n^{(a)}(-1)]\, C_n^{(a)}(x) \\ &\quad - N(-1)^n\, C_n^{(a)}(0)\, C_n^{(a)}(x-1) \end{aligned}$$

and substitute this in the difference equation (1). Then we find by means of the form (7) of the difference equation for the classical Charlier polynomials

$$\begin{aligned}N[1+N(-1)^n\, C_n^{(a)}(-1)] &\sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x)\\ &-N^2(-1)^n\, C_n^{(a)}(0) \sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x-1)\\ &-N(-1)^n\, C_n^{(a)}(0)[aC_n^{(a)}(x)+(n-a-x)\\ &\times C_n^{(a)}(x-1)+xC_n^{(a)}(x-2)]=0.\end{aligned}$$

By using (4), (6), and the difference equation (7) we obtain

$$aC_n^{(a)}(x)+(n-a-x)\, C_n^{(a)}(x-1)+xC_n^{(a)}(x-2)=-C_{n-1}^{(a)}(x-2).$$

Hence

$$\begin{aligned}N[1+N(-1)^n\, C_n^{(a)}(-1)] &\sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x)\\ &-N^2(-1)^n\, C_n^{(a)}(0) \sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x-1)\\ &+N(-1)^n\, C_n^{(a)}(0) C_{n-1}^{(a)}(x-2)=0.\end{aligned}$$

This formula must be valid for all values of $a>0$ and $N>0$. The left-hand side is a polynomial in N. So each coefficient of this polynomial has to be zero. This implies that

$$C_n^{(a)}(-1) \sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x)-C_n^{(a)}(0) \sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x-1)=0 \tag{13}$$

and

$$\sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x)+(-1)^n\, C_n^{(a)}(0) C_{n-1}^{(a)}(x-2)=0.$$

This can be simplified to

$$\sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x)=(-1)^{n-1}\, C_n^{(a)}(0)\, C_{n-1}^{(a)}(x-2) \tag{14}$$

and

$$\sum_{i=0}^{\infty} A_i(x)\, \Delta^i C_n^{(a)}(x-1) = (-1)^{n-1}\, C_n^{(a)}(-1)\, C_{n-1}^{(a)}(x-2), \tag{15}$$

since $C_n^{(a)}(0) \neq 0$ and $C_n^{(a)}(-1) \neq 0$.

We will show that (14) and (15) have a unique solution for the coefficients $\{A_i(x)\}_{i=0}^{\infty}$ which gives rise to the following theorem.

THEOREM 1. *The generalized Charlier polynomials* $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ *satisfy a unique difference equation of the form* (1), *where*

$$A_0(x) := A_0(n, a) = (-1)^{n-1}\, C_{n-1}^{(a)}(-2), \qquad n = 0, 1, 2, \ldots, \tag{16}$$

and

$$\begin{aligned} A_i(x) &:= A_i(a, x) \\ &= \sum_{k=1}^{i} (-1)^k\, C_{i-k}^{(-a)}(-x+1) \\ &\quad \times [C_k^{(a)}(-1)\, C_k^{(a)}(x-2) - C_k^{(a)}(-2)\, C_k^{(a)}(x-1)], \qquad i = 1, 2, 3, \ldots. \end{aligned} \tag{17}$$

The proof of this theorem can be found in the next section. Here Formula (8) is important. Formula (8) can be stated in other words as follows. If we define the matrix $T := (t_{ij})_{i,j=1}^{n}$ $(n \geq 1)$ with entries

$$t_{ij} := \begin{cases} C_{i-j}^{(a)}(x), & j \leq i, \\ 0, & j > i, \end{cases}$$

then this matrix T is a triangular matrix with determinant 1 and the inverse U of this matrix is given by $T^{-1} := U = (u_{ij})_{i,j=1}^{n}$ with entries

$$u_{ij} := \begin{cases} C_{i-j}^{(-a)}(-x), & j \leq i, \\ 0, & j > i. \end{cases}$$

The difference equation given by (1), (16), and (17) is of infinite order for all values of the parameter a. This can be seen as follows. From (17) it is clear that degree $[A_i(x)] \leq i$ for all $i = 1, 2, 3, \ldots$. Now we compute the coefficient h_i of x^i in the polynomial $A_i(x)$. By using the definition (2) we easily see that

$$C_n^{(a)}(x) = \frac{1}{n!}\, x^n + \text{lower order terms}.$$

Hence, from (17) we find for $i=1,2,3,\ldots$ by using (10) and (6)

$$\begin{aligned}h_i &= \sum_{k=1}^{i}(-1)^k\frac{(-1)^{i-k}}{(i-k)!}\left[\frac{C_k^{(a)}(-1)}{k!}-\frac{C_k^{(a)}(-2)}{k!}\right]\\ &=\frac{(-1)^i}{i!}\left[\sum_{k=0}^{i}\binom{i}{k}C_k^{(a)}(-1)-\sum_{k=0}^{i}\binom{i}{k}C_k^{(a)}(-2)\right]\\ &=\frac{(-1)^i}{i!}\left[C_i^{(a)}(i-1)-C_i^{(a)}(i-2)\right]=\frac{(-1)^i}{i!}C_{i-1}^{(a)}(i-2).\end{aligned}$$

This shows that the difference equation given by (1), (16), and (17) is of infinite order, since we have by using (2) and (3)

$$C_{i-1}^{(a)}(i-2)=L_{i-1}^{(-1)}(a),\qquad i=1,2,3,\ldots,$$

or

$$C_{i-1}^{(a)}(i-2)=-\frac{a}{i-1}C_{i-2}^{(a)}(i-1)=-\frac{a}{i-1}L_{i-2}^{(1)}(a),\qquad i=2,3,4,\ldots.$$

Only when a is a zero of some Laguerre polynomial might one of the leading coefficients h_i be zero (for some value of i), but in that case we have $h_{i+1}\neq 0$, since two consecutive Laguerre polynomials have interlacing zeros.

Moreover, the following theorem shows that the infinite order is unavoidable.

THEOREM 2. *Every linear difference equation of the form*

$$\begin{aligned}N\sum_{i=0}^{\infty}B_i(x)\,\Delta^{k_i}\nabla^{i-k_i}y(x)&+x\Delta\nabla y(x)\\ &+(a-x)\,\Delta y(x)+ny(x)=0,\qquad 0\leqslant k_i\leqslant i,\end{aligned}$$

satisfied by the polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ *has infinite order.*

Proof. From (4), (5), and (6) we easily find that

$$\nabla C_n^{(a)}(x)=C_{i-1}^{(a)}(x-1),\qquad n=0,1,2,\ldots.$$

This implies, in view of (6), that the leading coefficient of $\Delta^{k_i}\nabla^{i-k_i}C_n^{(a)}(x)$ is equal to that of $\Delta^iC_n^{(a)}(x)$. This implies, in view of the relations (14) and (15), that the leading coefficient of each $B_i(x)$ equals that of each corresponding $A_i(x)$. This proves Theorem 2.

Finally we refer to Section 6 for more results concerning the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ of the difference equation (1).

5. Proof of Theorem 1

In this section we will prove that (14) and (15) have a unique solution for the coefficients $\{A_i(x)\}_{i=0}^{\infty}$ with $\{A_i(x)\}_{i=1}^{\infty}$ independent of n and $A_0(x)$ independent of x.

Moreover, we will show that this unique solution is given by (16) and (17).

Formula (13) can be written as

$$\sum_{i=0}^{\infty} A_i(x)[C_n^{(a)}(-1)\,\Delta^i C_n^{(a)}(x) - C_n^{(a)}(0)\,\Delta^i C_n^{(a)}(x-1)] = 0.$$

Hence

$$\sum_{i=1}^{\infty} A_i(a,x)[C_n^{(a)}(-1)\,\Delta^i C_n^{(a)}(x) - C_n^{(a)}(0)\,\Delta^i C_n^{(a)}(x-1)]$$
$$= A_0(n,a)[C_n^{(a)}(0)\,C_n^{(a)}(x-1) - C_n^{(a)}(-1)\,C_n^{(a)}(x)].$$

The right-hand side vanishes for $x=0$ and since this must be valid for all values of n and $a>0$ we conclude step by step that $A_i(0) = A_i(a,0) = 0$ for all $i = 1, 2, 3, \ldots$.

Now (16) easily follows from (14) or (15), since $C_n^{(a)}(0) \neq 0$ and $C_n^{(a)}(-1) \neq 0$.

By using (14), (15), (16), and (6) we obtain

$$\sum_{i=1}^{n} A_i(a,x)\,C_{n-i}^{(a)}(x-1)$$
$$= (-1)^{n-1}\,C_n^{(a)}(-1)\,C_{n-1}^{(a)}(x-2) - (-1)^{n-1}\,C_n^{(a)}(x-1)\,C_{n-1}^{(a)}(-2)$$
$$= (-1)^n\,[C_n^{(a)}(-1)\,C_n^{(a)}(x-2) - C_n^{(a)}(-2)\,C_n^{(a)}(x-1)].$$

Now we use Formula (8) to obtain (17).

Finally we show that (14) must have the same solution. Since we have

$$C_n^{(a)}(x) = \Delta C_n^{(a)}(x-1) + C_n^{(a)}(x-1)$$

we find for $n \geqslant 1$, by using (6), that

$$\sum_{i=1}^{n} A_i(x)\,\Delta^i C_n^{(a)}(x) = \sum_{i=1}^{n} A_i(x)\,\Delta^{i+1} C_n^{(a)}(x-1) + \sum_{i=1}^{n} A_i(x)\,\Delta^i C_n^{(a)}(x-1)$$
$$= \sum_{i=1}^{n-1} A_i(x)\,\Delta^i C_{n-1}^{(a)}(x-1) + \sum_{i=1}^{n} A_i(x)\,\Delta^i C_n^{(a)}(x-1).$$

Since the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n it is sufficient to show that for $n \geqslant 1$

$$(-1)^{n-1}\left[C_{n-1}^{(a)}(-1)\,C_{n-1}^{(a)}(x-2) - C_{n-1}^{(a)}(-2)\,C_{n-1}^{(a)}(x-1)\right]$$
$$+(-1)^n\left[C_n^{(a)}(-1)\,C_n^{(a)}(x-2) - C_n^{(a)}(-2)\,C_n^{(a)}(x-1)\right]$$

equals

$$(-1)^{n-1}\,C_n^{(a)}(0)\,C_{n-1}^{(a)}(x-2) - (-1)^{n-1}\,C_n^{(a)}(x)\,C_{n-1}^{(a)}(-2).$$

The proof of this is straightforward and follows by using the fact that

$$C_{n-1}^{(a)}(x) = \Delta C_n^{(a)}(x) = C_n^{(a)}(x+1) - C_n^{(a)}(x).$$

6. Some Remarks

We have proved that the polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ satisfy a unique difference equation of the form (1) and that the coefficients $\{A_i(x)\}_{i=0}^{\infty}$ are given by (16) and (17). In Section 4 we already showed that this difference equation is of infinite order since

$$A_i(x) = \frac{(-1)^i}{i!}\,C_{i-1}^{(a)}(i-2)\,x^i + \text{lower order terms}, \qquad i = 1, 2, 3, \ldots.$$

Some more details about the coefficients can easily be discovered. For instance, note that the coefficients are both polynomials in x and in a. As a polynomial in x the coefficient $A_i(x)$ usually has degree i. Moreover, if degree$[A_i(x)] < i$ then we have degree$[A_{i+1}(x)] = i+1$. As a polynomial in a the coefficient $A_i(x)$ has degree $2i-2$. Moreover, we have by straightforward calculations

$$A_i(x) = \frac{(-1)^i\,x}{i!\,(i-1)!}\,a^{2i-2} + \text{lower order terms}, \qquad i = 1, 2, 3, \ldots.$$

The classical Charlier polynomials also satisfy a difference equation of infinite order. This can be shown as follows. Similar to Formula (9) we have more general

$$y(x-1) = \sum_{i=0}^{\infty} (-1)^i\,\Delta^i y(x)$$

for polynomials $y(x)$. This implies, by using (4) and (7), that the classical Charlier polynomials satisfy the infinite order difference equation given by

$$x \sum_{i=1}^{\infty} (-1)^i \Delta^i y(x) + a\,\Delta y(x) + ny(x) = 0, \qquad y(x) = C_n^{(a)}(x).$$

So the difference equation (1) for the generalized Charlier polynomials $\{C_n^{a,N}(x)\}_{n=0}^{\infty}$ can also be written in the form

$$N \sum_{i=0}^{\infty} A_i(x)\,\Delta^i y(x) + x \sum_{i=1}^{\infty} (-1)^i \Delta^i y(x) + a\,\Delta y(x) + ny(x) = 0.$$

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References

1. H. Bavinck, "On Polynomials Orthogonal with Respect to an Inner Product Involving Differences," Report 92-81, Delft University of Technology, Faculty of Technical Mathamatics and Informatics, 1992; *J. Comput. Appl. Math.*, to appear.
2. H. Bavinck, "A Difference Operator of Infinite Order with Sobolev-Type Charlier Polynomials as Eigenfunctions," Report 94-01, Delft University of Technology, Faculty of Technical Mathematics and Informatics, 1994; submitted for publication.
3. H. Bavinck and H. van Haeringen, Difference equations for generalized Meixner polynomials, *J. Math. Anal. Appl.* **184** (1994), 453–463.
4. C. Brezinski, L. Gori, and A. Ronveaux (Eds.), Orthogonal polynomials and their applications, *in* "IMACS Annals on Computing and Applied Mathematics," Vol. 9, p. 418, J. C. Baltzer AG, Basel, 1991.
5. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Mathematics and Its Applications, Vol. 13, Gordon & Breach, New York, 1978.
6. J. Koekoek and R. Koekoek, On a differential equation for Koornwinder's generalized Laguerre polynomials, *Proc. Amer. Math. Soc.* **112** (1991), 1045–1054.
7. R. Koekoek, Differential equations for symmetric generalized ultraspherical polynomials, *Trans. Amer. Math. Soc.* **345** (1994), 47–72.
8. T. H. Koornwinder, Orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta} + M\delta(x+1) + N\delta(x-1)$, *Canad. Math. Bull.* **27**(2) (1984), 205–214.